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THE COMPARATIVE STATICS OF CONSTRAINED OPTIMIZATION PROBLEMS

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This paper develops and applies some new results in the theory of monotone comparative statics. Let f be a real-valued function defined on R^l and consider the problem of maximizing $f(x)$ when x is constrained to lie in some subset C of R^l. We develop a natural way to order the constraint sets C and find the corresponding restrictions on the objective function f that guarantee that optimal solutions increase with the constraint set. We apply our techniques to problems in consumer, producer, and portfolio theory. We also use them to generalize Rybcsynski's theorem and the LeChatelier principle.

KEYWORDS: Lattices, concavity, supermodularity, comparative statics, LeChatelier principle, Rybcsynski's theorem, normality.

1. INTRODUCTION

IN RECENT YEARS, the methods of lattice programming have been used widely and successfully to deal with comparative statics problems that arise in optimization or game theoretic models.[2] In some cases, these methods have solved problems that were previously intractable; in other cases, they have provided more transparent (and thus more instructive) proofs to basically known results. The broad success of these methods highlights the fact that seemingly different comparative statics problems in economic theory often have a very similar mathematical structure.

There are broadly two types of comparative statics problems to which lattice programming methods have been applied, which we shall refer to as types A and B throughout this paper. Type A problems concern the change in the solution to a maximization problem as the objective function changes; type B problems concern the change in the solution to a maximization problem when the constraint set changes. This paper focuses on the latter; we consider those type B problems that the currently available techniques, in particular, those developed in Topkis (1978, 1998) and Milgrom and Shannon (1994), cannot address

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[2]For a textbook introduction to these methods, see Topkis (1998).

with complete success. Our approach is related to theirs in a natural way and shares some of its important features: the arguments are simple and geometric, and "regularity" assumptions (like the continuity of objective functions or the uniqueness and continuity of optimal solutions) are either absent or kept to a minimum. We apply our techniques to solve some central problems in demand, producer, and portfolio theory. We also use them to generalize Rybcsynski's theorem and the LeChatelier principle.

We first give a simplified version of the standard comparative statics theorem applicable to type B problems. This theorem is usually stated in the general context of a lattice $(X, \geq)$, but we shall be focussing in this paper on the case where X is a subset of the standard Euclidean space R^l and the order $\geq$ is the standard product order, i.e., $y \geq x$ if $y_i \geq x_i$ for $i = 1, 2, \ldots, l$. In this case, the supremum of two points x and y, denoted $x \vee y$, is given by $(x \vee y)_i = \max\{x_i, y_i\}$ and their infimum, $x \wedge y$, is given by $(x \wedge y)_i = \min\{x_i, y_i\}$. We require the objective function $f : X \to R$ to be *supermodular*; when f is sufficiently differentiable (specifically, C^2), this is equivalent to $\partial^2 f/\partial x_i \partial x_j \geq 0$ for all $i \neq j$. This condition is usually interpreted as a type of complementarity. For example, when f is a production function, it means that the marginal productivity of j, which is $\partial f/\partial x_j$, increases with x_i.

Let S and S' be two possible constraint sets. We are interested in comparing $\arg\max_{x \in S'} f(x)$ and $\arg\max_{x \in S} f(x)$. To keep our discussion simple, assume that in both cases, the optimal solution exists and is unique. Assuming that f is supermodular, what condition on S' and S is sufficient to guarantee that $\arg\max_{x \in S'} f(x) \geq \arg\max_{x \in S} f(x)$? It is known that a sufficient condition is for S' to dominate S in the *strong set order* (see Topkis (1978) and Milgrom and Shannon (1994)).[3] The set S' dominates S in the strong set order if for any y in S' and x in S, their supremum $(x \vee y)$ is in S' and their infimum $(x \wedge y)$ is in S.

The objective of this paper is to consider comparative statics problems where the standard theorem just stated cannot be easily applied; specifically, we focus on problems where the change in the constraint set is not comparable in the strong set order. To get around this difficulty, we introduce a new way to order sets, which we call the *$\mathcal{C}$-flexible set order*. This is weaker than the strong set order in the sense that a set can dominate another in this order but not in the strong set order. We show that when S' dominates S in the $\mathcal{C}$-flexible set order, we can still conclude that $\arg\max_{x \in S'} f(x) \geq \arg\max_{x \in S} f(x)$ provided stronger assumptions are imposed on the objective function; a sufficient condition is that it must be both supermodular and concave.

To show the usefulness of our techniques, the next section gives a simple application to the theory of consumer demand. We define the $\mathcal{C}$-flexible set order and explain why the change in the budget set following an increase in income is comparable in this weaker order (but not in the strong set order).

[3]A fuller statement of this fundamental result is presented in Section 3.

Thus our techniques are applicable and we can use them to obtain conditions for normal demand.

Following from that, Section 3 gives a fuller account of the principal comparative statics result for type B problems. We generalize the result slightly to suit our needs. Section 4 develops the main comparative statics results of this paper. Finally, Section 5 provides more examples where the constraint set change is comparable in the $\mathcal{C}$-flexible set order but not in the strong set order: we show how our techniques can be applied to these problems. Although this paper focusses on theory and applications in higher dimensions, the two-dimensional case is particularly intuitive and also has interesting applications. We consider this in the supplement (Quah (2007)).

2. MOTIVATING EXAMPLES

In this section we will give two examples of comparative statics problems that cannot be easily addressed using the standard lattice programming results, but which can be solved using our methods.

Because we are interested in comparing optimal solutions, and these may be sets rather than singletons, we first need an intuitive way to compare sets. Let S and S' be two sets in R^l. We say that S' is *i-higher* than S if, whenever both sets are nonempty, for any x in S there is x' in S' such that $x'_i \geq x_i$ and for any x' in S' there is x in S such that $x'_i \geq x_i$. We say that S' is *higher* than S if, whenever both sets are nonempty, for any x in S there is x' in S' such that $x' \geq x$ and for any x' in S' there is x in S such that $x' \geq x$.[4]

EXAMPLE 1: Consider a consumer who maximizes a utility function $U: X \to R$, where $X = R^l_{++}$ or R^l_+, while facing a budget constraint. At price p in R^l_+ and income $w > 0$, we denote his budget set by $B(p, w)$, where $B(p, w) = \{x \in X : p \cdot x \leq w\}$. We refer to $D(p, w) = \arg\max_{x \in B(p,w)} U(x)$ as *the demand set at* (p, w). If the demand set is nonempty and unique at every (p, w) in $R^l_{++} \times R_{++}$, then the map from (p, w) to $D(p, w)$ will be referred to as the *demand function*. Demand is said to be *i-normal* if $D(p, w')$ is i-higher than $D(p, w)$ whenever $w' > w$; demand is *normal* if $D(p, w')$ is higher than $D(p, w)$ whenever $w' > w$. If the agent has a demand function, then i-normality (normality) means that demand for good i (all goods) increases with income. We are interested in conditions under which either form of normality holds.[5]

[4] Throughout this paper, when we say that something is higher or increasing, we mean that it is higher or increasing in the *weak* sense. Most of the inequalities in this paper are weak, so this convention is natural. When we want an inequality to be strict, we will say so explicitly, as in strictly higher, strictly increasing, etc.

[5] Normality plays an important role in economic theory. For example, it is well known that i-normality is sufficient to guarantee that demand for i is a decreasing function of its own price, i.e., the demand curve for i is downward sloping. Normality of one form or another is also important in general equilibrium comparative statics (see Nachbar (2002) and Quah (2003)).

To answer our question, we could first try to use the standard comparative statics theorem. *If* whenever $w' > w$, we have $B(p, w')$ dominating $B(p, w)$ in the strong set order, then an application of the standard theorem (as stated in the Introduction) will tell us that $D(p, w')$ is higher than $D(p, w)$ if U is supermodular. However, $B(p, w')$ does not dominate $B(p, w)$: to see this note that for any y in $B(p, w')$ and x in $B(p, w)$, the bundle $x \wedge y$ is indeed in $B(p, w)$ because $p \cdot (x \wedge y) \leq w$, but $x \vee y$ need not be in $B(p, w')$ because $p \cdot (x \wedge y)$ may be strictly greater than w'. So we cannot apply the standard theorem.

In contrast, it is not hard to see that $B(p, w')$ *does* dominate $B(p, w)$ in the *$\mathcal{C}$-flexible set order*. A set S' is said to dominate S in this order if for any x in S and y in S', with x and y unordered (see Figure 1), there are points a in S and b in S' such that x, a, y, and b form a backward-bending parallelogram. Formally, there is a scalar λ in [0, 1] such that $a = x \wedge y + \lambda v$ and $b = x \vee y - \lambda v$, where $v = y - x \wedge y$. (Note that the points x, y, $x \vee y$, and $x \wedge y$ form a rectangle in the following sense: $y - x \wedge y = x \vee y - x$, $x - x \wedge y = x \vee y - y$, and the two vectors are orthogonal.)

To guarantee normality, the supermodularity of U alone is not enough, but the addition of concavity assumptions on U, together with supermodularity, will give us the desired conclusion. For any function f defined on the convex subset X of R^l, we say that f is *concave in direction* $v \neq 0$ if, for all x, the map from the scalar s to $f(x + sv)$ is concave. (The domain of this map is taken to be the largest interval such that $x + sv$ lies in X.) We say that f is *i-concave* if it is concave in direction v for any $v > 0$ with $v_i = 0$. We say that f is *partially concave* if it is i-concave for all i. For example, assuming $X = R^2_{++}$, the function given by $F(x_1, x_2) = x_1 x_2^2$ is 2-concave (because it is a concave function of x_1) but not 1-concave (because it is a convex function of x_2). Obviously, if f is concave, then it is partially concave, but the converse does not hold. For example, the function $F(x_1, x_2) = x_1 x_2$ is partially concave but not concave.

PROPOSITION 1: (i) *Suppose that $U : X \to R$ (where X is either R^l_+ or R^l_{++}) is supermodular and i-concave. Then the demand it generates is i-normal.* (ii) *If U is supermodular and partially concave, then it generates normal demand.*

We postpone the proof of Proposition 1 to Section 3. As a simple application of this result, consider the additive utility function $U(x) = \sum_{k=1}^{l} u_k(x_k)$. This function is clearly supermodular. If for all $k \neq i$, u_k is concave, then U is i-concave, so we know that it generates i-normal demand. If u_k is concave for all k, then U is concave and, in particular, partially concave, so it generates normal demand. Suppose we interpret the goods to be contingent commodities in l different states of the world. Then the von Neumann–Morgenstern axioms guarantee that the preference over contingent consumption can be evaluated via expected utility, so that U will indeed be additive. Thus, the demand for contingent consumption will be normal if markets are complete (which guarantees that the commodity space is R^l_+ and not some lower dimensional subset).

A much studied alternative to expected utility is Choquet expected utility. Provided the capacity is convex, the agent's utility function takes the form $\bar{U}(x) = \min_{\mu \in C(\nu)}[\sum_{k=1}^{l} \mu_k u(x_k)]$, where $u : R_+ \to R$, ν is the convex capacity and $C(\nu)$ is the core of ν. It is straightforward to check that $\bar{U}$ is concave if u is concave. It is less straightforward to check, but still true, that if u is increasing, $\bar{U}$ will be supermodular (see Marinacci and Montrucchio (2004, Theorem 35)). So in this case, $\bar{U}$ generates normal demand for contingent consumption.

It is worth pointing out that Proposition 1 does not make many of the assumptions that are typically made to obtain comparative statics results in demand theory. Among other things, we have not assumed that U is C^2, increasing, or strictly quasiconcave because none of these is essential to the basic result. If we make some of these assumptions, it would be possible to sharpen the conclusion a little. For example, suppose that in addition to being supermodular and partially concave, U is also *strictly* quasiconcave. Strict quasiconcavity guarantees that demand is always unique, so for any $w' > w > 0$, Proposition 1 tells us that $D(p, w') \geq D(p, w)$. (Note that $D(p, w')$ and $D(p, w)$ are now vectors.) If, in addition, U is increasing, then demand always obeys the budget identity, so that $D(p, w') > D(p, w)$.[6]

EXAMPLE 2: This is an application of Proposition 1 in a production context. We wish to determine the impact of a change in input price on a firm's optimal output. Suppose the firm produces a single output using l inputs, with the production function $F : R^l_{++} \to R$. To keep our discussion short, we shall make the standard assumptions, which guarantee that at each input price vector p in R^l_{++} and output $q > 0$, the cost-minimizing bundle

$$X(p, q) \equiv \underset{\{x \in R^l_{++} : F(x) \geq q\}}{\arg\min} \quad p \cdot x$$

exists and is unique. Furthermore, $X(p, q)$ coincides with the bundle maximizing output when input prices are at p and expenditure is kept at $p \cdot X(p, q)$.

Formally, the last property says that $X(p, q) = \arg\max_{x \in B(p, p \cdot X(p,q))} F(x)$, so we can say something about the comparative statics of $X(p, q)$ by appealing to Proposition 1, which tells us that $X_i(p, q)$ increases with q if F is supermodular and i-concave. Supermodularity in this context has a straightforward interpretation. It means that the inputs are complements in the production process: the added output from a given increase in factor i is higher when there is more of

[6]It is conventional to write $x > y$ if $x \geq y$ and $x \neq y$. The notation $x \gg y$ means that $x_i > y_i$ for all i.

factor $j \neq i$. The i-concavity of F says that when all inputs but i are increased along rays emanating from any input bundle x, the marginal increase in output is diminishing. In other words, as long as the level of input i is held fixed, increasing the input of other factors has a diminishing impact on output. This assumption is just a multidimensional version of the familiar law of diminishing (strictly speaking, in our case, nonincreasing) marginal returns and appears to be reasonable in many contexts. Notice also that we do not require F to be concave; this means, in particular, that F *can* exhibit increasing returns to scale. (Recall the example given earlier: $F(x_1, x_2) = x_1 x_2$.)[7]

What is the significance of $X_i(p, q)$ increasing with q? Assume that when $q > 0$ is produced, the firm derives from it a revenue of $V(q)$. The firm chooses q to maximize profit, which is $V(q) - C(p, q)$, where $C(p, q) = p \cdot X(p, q)$ is the cost function. Suppose we are interested in the reaction of q to a change in p_i; this is a type A comparative statics problem, and a standard result for these problems says that the optimal q decreases with p_i if C is supermodular in (p_i, q) (see Topkis (1998, Theorem 2.8.1)). This is in turn equivalent to $X_i(p, q)$ increasing with q, because by the envelope theorem, $\partial C/\partial p_i(p, q) = X_i(p, q)$.[8]

To recap, we have established the following comparative statics result: *if F is i-concave and supermodular, then output decreases with the price of i.* Clearly, if we want output to be decreasing with respect to *all* input prices, then a sufficient condition is that F is supermodular and partially concave.

It is worth asking whether there are alternative ways to obtain the same conclusion. One fairly obvious approach is the following. Suppose $V \circ F$ is a supermodular function, which guarantees that the profit function $V \circ F(x) - \sum_{i=1}^{l} p_i x_i$ is also supermodular. The behavior of optimal input as p_i changes is a type A comparative statics problem, and the supermodularity of the profit function guarantees that an increase in p_i reduces the (optimal) use of *all* inputs (see Topkis (1998, Theorem 2.8.1)). It follows that output also falls.

The difference between the approach using Proposition 1 and the one just described is that the former makes no assumptions about the revenue function: all its assumptions are imposed on the production function. In particular, its assumption that F is supermodular does not, as a rule, imply that $V \circ F$ is supermodular. (An exception being the case where the firm is a price-taker, so V is linear in q.)

[7]More generally, a Cobb–Douglas production function, which has the form $F(x) = \bigtimes_{j=1}^{l} x_j^{\alpha_j}$, where $\alpha_j > 0$ for all j, will be i-concave if and only if $\sum_{j \neq i} \alpha_j \leq 1$, and will be concave if and only if $\sum_{j=1}^{l} \alpha^j \leq 1$. It has strictly diminishing returns to scale if $\sum_{j=1}^{l} \alpha^j < 1$, constant returns to scale if $\sum_{j=1}^{l} \alpha^j = 1$, and strictly increasing returns to scale if $\sum_{j=1}^{l} \alpha^j > 1$. The function is always supermodular.

[8]This connection between normality of input demand and the impact on marginal costs (and thus output) of an input price change is well known (see McFadden (1978) and Athey, Milgrom, and Roberts (1998)).

Proposition 1 is a special case of the general comparative statics results that we shall develop in the next two sections of the paper. It is worth emphasizing that the budget set change considered in this proposition is just *one* example of a constraint set change that is comparable in the $\mathcal{C}$-flexible set order. The ordering is sufficiently "versatile" to accommodate other types of constraint set changes; these are considered in Section 4 (Proposition 4) and in some of the examples in Section 5.[9]

3. THE STANDARD THEORY

In this section, we introduce the fundamental result in the theory of monotone comparative statics for type B problems. Our treatment is slightly different from the standard one (see Topkis (1978, 1998) and Milgrom and Shannon (1994)) because we will present the material in a nonlattice context. This variation is minor in the sense that it does not make the main comparative statics theorems (or at least the one we highlight in this section) any harder to prove, but the more general context is crucial for our purposes.[10]

Let X be a set and let $\triangledown$ and $\vartriangle$ be two operations on X, i.e., $\triangledown$ and $\vartriangle$ are maps from $X \times X$ to X, with the image of (x, y) under these maps denoted by $x \triangledown y$ and $x \vartriangle y$, respectively. We call a function $f : X \to R$ $(\triangledown, \vartriangle)$-*supermodular* if

$$f(x \triangledown y) - f(y) \geq f(x) - f(x \vartriangle y) \quad \text{for all } x \text{ and } y \text{ in } X. \tag{1}$$

The function f is said to be $(\triangledown, \vartriangle)$-*quasisupermodular* if for any x and y in X,

$$f(x) \geq (>)\, f(x \vartriangle y) \quad \Longrightarrow \quad f(x \triangledown y) \geq (>)\, f(y). \tag{2}$$

Clearly, a function that is $(\triangledown, \vartriangle)$-supermodular is also $(\triangledown, \vartriangle)$-quasisupermodular. The latter is an ordinal property in the sense that if f is $(\triangledown, \vartriangle)$-quasisupermodular, then so is $h \circ f$ for any strictly increasing function $h : R \to R$. Again, this is obvious.

Let S' and S be two sets in X. We say that S' dominates S in the *strong set order induced by* $(\triangledown, \vartriangle)$ (and write $S' \geq_{(\triangledown, \vartriangle)} S$) if for any x in S and y in S', $x \triangledown y$ is in S' and $x \vartriangle y$ is in S.

Theorem 1, which follows, says that the $(\triangledown, \vartriangle)$-quasisupermodularity of the objective function is necessary and sufficient for the solution set to a maximization problem to increase as the constraint set increases. (By increase, we mean that it becomes dominant in the strong set order induced by $(\triangledown, \vartriangle)$.) Theorem 1 is a slight generalization of a similar result found in Milgrom and Shannon (1994), in which X is assumed to be a lattice, and $\triangledown$ and $\vartriangle$ are the

[9]See also the supplement (Quah (2007)) and Quah (2006).
[10]See Ruble (2004) for a related generalization.

supremum and infimum operations; it can also be proven by a superficial modification of their proof. Nonetheless, we shall give the proof of the "only if" part of this result because we need to appeal to it in a crucial way later.

THEOREM 1: *Let $\triangledown$ and $\triangle$ be two operations on X. Then $f: X \to R$ is a $(\triangledown, \triangle)$-quasisupermodular function if and only if it obeys the property that whenever $S' \geq_{(\triangledown,\triangle)} S$ for subsets S' and S of X, then*

$$\arg\max_{x \in S'} f(x) \geq_{(\triangledown,\triangle)} \arg\max_{x \in S} f(x).$$

For the proof, see Appendix A.

4. CONSTRAINED OPTIMIZATION PROBLEMS IN R^l

Our goal in this section is to establish a comparative statics result that has Proposition 1 (in Section 2) as a special case. We divide our discussion into four parts. In Section 4.1, we focus on the conditions imposed on the objective functions in our theory. The new property we require is $\mathcal{C}$-supermodularity (or its ordinal analog, $\mathcal{C}$-quasisupermodularity). Recall that in Proposition 1 we assumed that the utility function is supermodular and partially concave: we will show that these conditions are sufficient for $\mathcal{C}$-supermodularity. In Section 4.2, we turn to the conditions on the constraint sets. We give a more thorough treatment of the $\mathcal{C}$-flexible set order that we introduced in Section 2 and identify some conditions under which constraint sets are comparable in this order. In Section 4.3, we state and prove our main comparative statics result, of which Proposition 1 is a corollary. Finally, Section 4.4 deals with the related literature and other issues.

We depart from the general setting of the last section and return to the setting of the Introduction and Section 2: our context is the Euclidean space R^l endowed with the product order $\geq$. We denote the supremum and infimum with respect to this order by $\vee$ and $\wedge$, respectively. A subset X of R^l is a *sublattice* (of R^l) if for x and y in X, both $x \vee y$ and $x \wedge y$ are in X. Following the definition in the last section, a function $f: X \to R$ is $(\vee, \wedge)$-supermodular, if

$$f(x \vee y) - f(y) \geq f(x) - f(x \wedge y) \quad \text{for any } x \text{ and } y \text{ in } X. \tag{3}$$

From now on, any function that satisfies (3) will simply be referred to as a supermodular function, without explicit reference to the product order. For our purposes, it is important that one has a good geometrical picture of supermodularity. When x and y are ordered, (3) holds trivially, so let us assume that they are not ordered. In that case, as we already pointed out in Section 2, the four points x, y, $x \vee y$, and $x \wedge y$ lie on a two-dimensional plane and form a rectangle (see Figure 1). The supermodularity of f requires that the difference in the function's value on the right side of the rectangle, $f(x \vee y) - f(y)$, be larger than the difference on the left side, which is $f(x) - f(x \wedge y)$.

For the comparative statics results we have in mind, we require the objective function to have a property that is stronger than supermodularity.

4.1. *$\mathcal{C}$-Supermodular Functions*

We now assume that the domain $X \subseteq R^l$, in addition to being a sublattice, is also a convex set. This assumption will be maintained throughout the paper. For λ in $[0, 1]$, the operation $\triangledown_i^\lambda$ on X is defined as follows: $x \triangledown_i^\lambda y = y$ if $x_i \le y_i$ and $x \triangledown_i^\lambda y = \lambda x + (1-\lambda)(x \vee y)$ if $x_i > y_i$. Note that $\lambda x + (1-\lambda)(x \vee y)$ is in X because X is a convex sublattice of R^l. The operation $\triangle_i^\lambda$ on X is defined as $x \triangle_i^\lambda y = x$ if $x_i \le y_i$ and $x \triangle_i^\lambda y = \lambda y + (1-\lambda)(x \wedge y)$ if $x_i > y_i$. Figure 1 shows the points $a = x \triangle_i^\lambda y = x \wedge y + \lambda v$ and $b = x \triangledown_i^\lambda y = x \vee y - \lambda v$ in the case when $x_i > y_i$ and the two points are unordered. (Recall that $v = y - x \wedge y = x \vee y - x$.) The four points x, y, $x \triangledown_i^\lambda y$, and $x \triangle_i^\lambda y$ form a backward-bending parallelogram.

Suppose that the function $f : X \to R$ is $(\triangledown_i^\lambda, \triangle_i^\lambda)$-supermodular for some λ in $[0, 1]$; this requires

$$f(x \triangledown_i^\lambda y) - f(y) \ge f(x) - f(x \triangle_i^\lambda y) \quad \text{for all } x \text{ and } y \text{ in } X. \tag{4}$$

For two points x and y, if $x_i \le y_i$ or if $x > y$, it is not hard to check that (4) holds trivially. The interesting case occurs when $x_i > y_i$, and x and y are unordered. Referring to Figure 1 again, this property requires the difference in the function's value along the right side of the parallelogram, $f(x \triangledown_i^\lambda y) - f(y)$ to be greater than the difference along the left side, which is $f(x) - f(x \triangle_i^\lambda y)$.

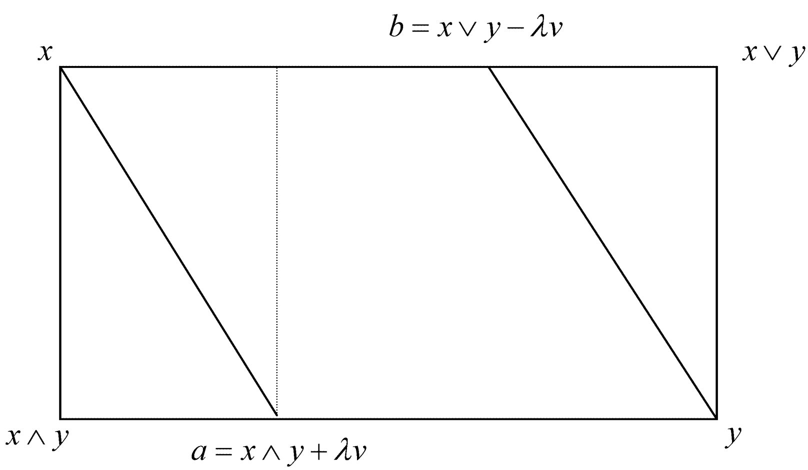

FIGURE 1.

We say that the function $f : X \to R$ is *$\mathcal{C}_i$-supermodular* if it is $(\triangledown_i^\lambda, \triangle_i^\lambda)$-supermodular for all $(\triangledown_i^\lambda, \triangle_i^\lambda)$ in the family $\mathcal{C}_i = \{(\triangledown_i^\lambda, \triangle_i^\lambda) : \lambda \in [0, 1]\}$. In other words, the "parallelogram inequality" (4) is required to hold for all the parallelograms formed as λ varies. Note that when $\lambda = 0$, (4) is just the "rectangular inequality" (3). Thus, it is clear that if f is $\mathcal{C}_i$-supermodular for all i (in which case we will refer to f as a *$\mathcal{C}$-supermodular* function), then it must also be supermodular.

Comparative statics results that rely on $\mathcal{C}$-supermodularity are only useful to the extent that we can show that this property holds under reasonable conditions. For this reason, our next result is important because it shows that $\mathcal{C}$-supermodularity arises from the marriage of two conditions that, in many situations, can both be expected to hold: one is supermodularity and the other is some version of concavity.

PROPOSITION 2: *The function $f : X \to R$ is $\mathcal{C}_i$-supermodular if it is supermodular and i-concave.*[11]

PROOF: Let x and y be two unordered elements in X, with $x_i > y_i$ (as in Figure 1). Define $v = x \vee y - x = y - x \wedge y$. Note that

$$f(x \triangledown_i^\lambda y) = f(\lambda x + (1 - \lambda)(x \vee y)) = f(x \vee y - \lambda v)$$

and, similarly, $f(x \triangle_i^\lambda y) = f(x \wedge y + \lambda v)$. We may decompose $f(x \vee y - \lambda v) - f(y)$ into $[f(x \vee y - \lambda v) - f(x \vee y)] + [f(x \vee y) - f(y)]$. We also have $v > 0$ and $v_i = 0$; by the fact that f is i-concave,[12]

$$\begin{aligned} &f(x \vee y - \lambda v) - f(x \vee y) \\ &\quad \geq f(x \vee y - \lambda v - (1 - \lambda)v) - f(x \vee y - (1 - \lambda)v) \\ &\quad = f(x) - f(x + \lambda v). \end{aligned}$$

Because f is supermodular, $f(x \vee y) - f(y) \geq f(x + \lambda v) - f(x \wedge y + \lambda v)$. Together these two inequalities imply that

$$\begin{aligned} &f(x \vee y - \lambda v) - f(y) \\ &\quad \geq [f(x) - f(x + \lambda v)] + [f(x + \lambda v) - f(x \wedge y + \lambda v)] \\ &\quad = f(x) - f(x \wedge y + \lambda v) \\ &\quad = f(x) - f(x \triangle_i^\lambda y) \end{aligned}$$

and we are done. *Q.E.D.*

[11] Recall the definition of i-concavity in Section 2.

[12] We are making use of the fact that when f is concave in direction v, we have $f(x - v) - f(x) \geq f(x - v - tv) - f(x - tv)$ for x in X and scalar $t > 0$.

Recall that, by definition, a function is partially concave if it is i-concave for all i. It follows from Proposition 2 that if f is supermodular and partially concave, then it is $\mathcal{C}$-supermodular. At this point it may be worth noting that the conditions imposed on U in Proposition 1 are precisely the ones that guarantee $\mathcal{C}_i$-supermodularity and $\mathcal{C}$-supermodularity.

A function $f:X \to R$ is *i-convex* (*partially convex*) if $-f$ is i-concave (partially concave). It is *submodular* (*$\mathcal{C}_i$-submodular*, *$\mathcal{C}$-submodular*) if $-f$ is supermodular ($\mathcal{C}_i$-supermodular, $\mathcal{C}$-supermodular). It follows immediately from Proposition 2 that a function $f:X \to R$ is $\mathcal{C}_i$-submodular if it is submodular and i-convex, and it is $\mathcal{C}$-submodular if it is submodular and partially convex.

It has been emphasized by Milgrom and Shannon (1994) in their influential study of comparative statics that comparative statics results rely on the ordinal, rather than the cardinal, properties of the objective function. For this reason we shall introduce an ordinal version of $\mathcal{C}$-supermodularity. We say that $f:X \to R$ is *$\mathcal{C}_i$-quasisupermodular* if it is $(\triangledown_i^\lambda, \triangle_i^\lambda)$-quasisupermodular for all $(\triangledown_i^\lambda, \triangle_i^\lambda)$ in $\mathcal{C}_i$. We say that f is $\mathcal{C}$-quasisupermodular if it is $\mathcal{C}_i$-quasisupermodular for all i. It is clear that any $\mathcal{C}_i$-supermodular function is also $\mathcal{C}_i$-quasisupermodular and that the latter is a strictly weaker property. Last, we call a function *$\mathcal{C}_i$-quasisubmodular* (*$\mathcal{C}$-quasisubmodular*) if $-f$ is $\mathcal{C}_i$-quasisupermodular ($\mathcal{C}$-quasisupermodular).

Suppose that $X = X_1 \times X_2$, where X_1 and X_2 are open intervals in R (so $X \subseteq R^2$). We assume that $f:X \to R$ is C^1, with $f_1 > 0$ and $f_2 > 0$, so that all the indifference curves are downward sloping.[13] What do the indifference curves of f look like if f is a $\mathcal{C}_2$-quasisupermodular function? If x_1 is represented on the horizontal axis and x_2 is represented on the vertical axis, then it is easy to see that $\mathcal{C}_2$-quasisupermodularity implies the following *declining slope property*: the slope of the indifference curve through (x_1, x_2) falls as x_1 increases while keeping x_2 fixed.[14] We can also perform a quick check to see that this property is implied by the conditions of Proposition 2. The slope of the indifference curve through (x_1, x_2) is $-f_1(x_1, x_2)/f_2(x_1, x_2)$. The derivative of this expression with respect to x_1 is

$$\frac{-f_2 f_{11} + f_{12} f_1}{f_2^2}.$$

This is nonnegative if $f_1 > 0$, $f_2 > 0$, and f is 2-concave (so $f_{11} \leq 0$) and supermodular (so $f_{12} \geq 0$).[15]

[13]Note that the subscripts denote partial derivatives. We shall use this notation whenever it is convenient and there is no danger of misinterpretation.

[14]Suppose not, so at (x_1^*, x_2^*) the indifference curve is downward sloping and strictly less steep than at (x_1^{**}, x_2^*), where $x_1^{**} > x_1^*$. Then for $\varepsilon > 0$ and sufficiently small, there is $\Delta > 0$ such that $f(x_1^* - \varepsilon, x_2^* + \Delta) = f(x_1^*, x_2^*)$ and $f(x_1^{**} - \varepsilon, x_2^* + \Delta) < f(x_1^{**}, x_2^*)$, which is a violation $\mathcal{C}_2$-quasisupermodularity.

[15]For the *sufficiency* of the declining slope property for $\mathcal{C}_2$-quasisupermodularity, see the supplement (Quah (2007)).

4.2. *The $\mathcal{C}$-Flexible Set Order*

Let S' and S be subsets of the convex sublattice X. We say that S' *is greater than (or dominates) S in the $\mathcal{C}_i$-flexible set order* (and write $S' \geq_i S$) if for any x in S and y in S', there is $(\triangledown_i^\lambda, \triangle_i^\lambda)$ in $\mathcal{C}_i$ such that $x \triangledown_i^\lambda y$ is in S' and $x \triangle_i^\lambda y$ is in S. Obviously, if S' dominates S in the strong set order induced by $(\triangledown_i^{\lambda^*}, \triangle_i^{\lambda^*})$ for some λ^* in $[0, 1]$, then S' i-dominates S in the $\mathcal{C}_i$-flexible set order. However, the latter concept is weaker (or more "flexible") because we allow λ to vary with every pair of x and y. We say that S' *is greater than (or dominates) S in the $\mathcal{C}$-flexible set order* (and write $S' \geq S$) if $S' \geq_i S$ for all i.

It is helpful to take a closer look at exactly what this set ordering requires. Suppose x is in S, y is in S', and $S' \geq_i S$. If $x_i \leq y_i$, then, for any λ in $[0, 1]$, $x \triangledown_i^\lambda y = y$ and $x \triangle_i^\lambda y = x$, so that the set ordering requirement is satisfied trivially. Thus, *to check that $S' \geq_i S$, we need only check the case where $x_i > y_i$.* If $x > y$, then for any λ, $x \triangledown_i^\lambda y = x$ and $x \triangle_i^\lambda y = y$ so the set ordering requires x to be in S' and y to be in S. If x and y are not ordered, we have the situation depicted in Figure 1. In this case, the $\mathcal{C}_i$-flexible set ordering requires that one can find two other points, a and b in Figure 1, such that b is in S' and a is in S, with the four points forming a backward-bending parallelogram.

The objective in a comparative statics problem is often to show that one optimal solution set is higher or i-higher than another (in the sense defined in Section 2). The next result relates this way to order sets with the $\mathcal{C}$-flexible set order.

PROPOSITION 3: *Let S' and S be nonempty subsets of a convex sublattice X in R^l.* (i) *If $S' \geq_i S$, then S' is i-higher than S.* (ii) *If $S' \geq S$, then S' is higher than S.*

PROOF: The proof of (i) is similar to (ii), so we shall prove only the latter. First assume that x is in S. We need to show that there is x' in S' such that $x' \geq x$. Given that S' is nonempty, there is y in S'. If $y \geq x$, set $x' = y$. If not, there is some j such that $x_j > y_j$. Because $S' \geq_j S$, there is λ such that $x \triangledown_j^\lambda y$ is in S'. Note that $x \vee y \geq x$, so $x \triangledown_j^\lambda y = \lambda x + (1 - \lambda)(x \vee y) \geq x$ and we can set $x' = x \triangledown_j^\lambda y$. We also need to check that for x' in S there is x in S such that $x \leq x'$, but we shall omit the similar proof. *Q.E.D.*

As a simple illustration, let $S' = \{(1 + t, 2), (2 + t, 1)\}$ and $S = \{(1, 2), (2, 1)\}$. For any $t > 0$ it is easy to see that $S' \geq_2 S'$, although for t in $(0, 1)$, $S' \ngeq_1 S$. We do have $S' \geq_1 S$ if $t \geq 1$, so in this case $S' \geq S$. Note that S' is not a superset of S, so a set can dominate another in this order without it being a superset of the other set.

The next result relates $\mathcal{C}$-quasisubmodularity with the $\mathcal{C}$-flexible set order and gives us a simple way to generate classes of ordered sets.

PROPOSITION 4: (i) *Suppose that* $G: X \to R$ *is a continuous, increasing, and* $\mathcal{C}_i$*-quasisubmodular function. Then*

$$G^{-1}((-\infty, k'']) \geq_i G^{-1}((-\infty, k']) \quad \text{whenever } k'' \geq k'. \tag{5}$$

(ii) *Suppose that* $G: X \to R$ *is continuous, strictly increasing, and obeys property* (5). *Then* G *is* $\mathcal{C}_i$*-quasisubmodular.*

Proposition 2 implies that if G is submodular and partially convex, i.e., i-convex for all i, then it is $\mathcal{C}_i$-submodular for all i (hence, $\mathcal{C}$-quasisubmodular). So an immediate corollary of Proposition 4(i) is that if $G: X \to R$ is continuous, increasing, submodular, and partially convex, then $G^{-1}((-\infty, k'']) \geq G^{-1}((-\infty, k'])$ whenever $k'' \geq k'$.

As a simple application of this result, consider the budget set $B(p, w)$ as defined in Example 1. Note that $B(p, w) = E^{-1}(w)$, where $E: X \to R$, given by $E(x) = p \cdot x$, is indeed continuous, increasing, submodular, and convex. Thus, $B(p, w') \geq B(p, w)$ whenever $w' > w$.

PROOF OF PROPOSITION 4: (i) Consider x in $G^{-1}((-\infty, k'])$ and y in $G^{-1}((-\infty, k''])$ with $x_i > y_i$. If y is also in $G^{-1}((-\infty, k'])$, we choose $\lambda = 1$, in which case $x \bigtriangledown_i^1 y = x$ and $x \bigtriangleup_i^1 y = y$. Clearly, the former is in $G^{-1}((-\infty, k''])$ and the latter is in $G^{-1}((-\infty, k'])$, as required. So assume that $G(y) > k'$. Note that $G(x \wedge y) \leq k'$ because G is increasing. The vector $v \equiv x \vee y - x = y - x \wedge y > 0$; because G is increasing and continuous, there is $\bar{\lambda}$ in $[0, 1]$ such that $G(x \wedge y + \bar{\lambda} v) = k'$. Thus $G(x) - G(x \wedge y + \bar{\lambda} v) \leq 0$. Because $x \wedge y + \bar{\lambda} v = x \bigtriangleup_i^{\bar{\lambda}} y$ and G is $\mathcal{C}_i$-quasisubmodular, we have $G(x \bigtriangledown_i^{\bar{\lambda}} y) \leq G(y)$, and thus $x \bigtriangledown_i^{\bar{\lambda}} y$ is in $G^{-1}((-\infty, k''])$.

(ii) Consider x and y, unordered, with $x_i > y_i$ and suppose that $G(x) = k'$ and $G(y) = k''$. Defining v as in the previous paragraph, the $\mathcal{C}_i$-quasisubmodularity of G is equivalent to, for any λ in $[0, 1]$,

$$G(x) \leq (<)\, G(x \wedge y + \lambda v) \quad \Longrightarrow \quad G(x \vee y - \lambda v) \leq (<)\, G(y). \tag{6}$$

If $k'' < k'$, then by the fact that G is increasing, $G(x \wedge y + \lambda v) \leq G(y) = k'' < k' = G(x)$, which means that (6) is vacuously true for λ in $[0, 1]$. If $k'' = k'$, then because G is strictly increasing, (6) is vacuously true for λ in $[0, 1)$ and trivially true at $\lambda = 1$. So we assume that $k'' > k'$; given that $G^{-1}((-\infty, k'']) >_i G^{-1}((-\infty, k'])$, we know that there is $\bar{\lambda}$ such that $x \wedge y + \bar{\lambda} v$ is in $G^{-1}((-\infty, k'])$ and $x \vee y - \bar{\lambda} v$ is in $G^{-1}((-\infty, k''])$. Given that G is continuous and increasing, there is $\lambda^* \geq \bar{\lambda}$ such that $G(x \wedge y + \lambda^* v) = k'$ and $G(x \vee y - \lambda^* v) \leq k''$. Furthermore, because G is strictly increasing, $G(x' \wedge y + \lambda v) < k'$ for $\lambda < \lambda^*$ and, for $\lambda > \lambda^*$, we have $G(x' \wedge y + \lambda v) > k'$ and $G(x' \vee y - \lambda v) < k''$. This means that (6) holds. *Q.E.D.*

4.3. *Comparative Statics*

Let X be a convex sublattice of R^l. We say that $F: X \to R$ has the *i-increasing property* if $\arg\max_{x\in S'} F(x) \geq_i \arg\max_{x\in S} F(x)$ whenever $S' \geq_i S$. We say that F has the *increasing property* if it is i-increasing for all i; in particular, this means that $\arg\max_{x\in S'} F(x) \geq \arg\max_{x\in S} F(x)$ whenever $S' \geq S$. If F has the i-increasing property, then Proposition 3(i) tells us that (a) whenever $S' \geq_i S$, the set $\arg\max_{x\in S'} F(x)$ is i-higher than $\arg\max_{x\in S} F(x)$. If F has the increasing property, then, in addition to (a), Proposition 3(ii) tells us that (b) whenever $S' \geq S$, the set $\arg\max_{x\in S'} F(x)$ is higher than $\arg\max_{x\in S} F(x)$.[16]

The main comparative statics result of this paper says that the i-increasing property is equivalent to the $\mathcal{C}_i$-quasisupermodularity of the objective function.

THEOREM 2: *The function $F: X \to R$ is $\mathcal{C}$-quasisupermodular* ($\mathcal{C}_i$*-quasisupermodular*) *if and only if it has the i-increasing property* (*increasing property*).

PROOF: The nonbracketed claim follows logically from the bracketed version, so we shall prove only the latter. We first prove necessity. Assume that $S' \geq_i S$, and let x' be in $\arg\max_{x\in S} F(x)$ and let y be in $\arg\max_{x\in S'} F(x)$ with $x'_i > y_i$ (recall that this is the only case that needs checking). Given that $S' \geq_i S$, there is λ^* such that $x' \bigtriangledown_i^{\lambda^*} y \neq y$ is in S' and $x' \bigtriangleup_i^{\lambda^*} y \neq x'$ is in S. It is clear that $C' \equiv \{y,\ x' \bigtriangledown_i^{\lambda^*} y\} \geq_{(\bigtriangledown_i^{\lambda^*}, \bigtriangleup_i^{\lambda^*})} \{x',\ x' \bigtriangleup_i^{\lambda^*} y\} \equiv C$. Obviously, x' is in $\arg\max_{x\in C} F(x)$ and y is in $\arg\max_{x\in C'} F(x)$. Because F is $\mathcal{C}_i$-quasisupermodular, in particular it is $(\bigtriangledown_i^{\lambda^*}, \bigtriangleup_i^{\lambda^*})$-quasisupermodular, by applying Theorem 1, $F(x' \bigtriangledown_i^{\lambda^*} y) = F(y)$ and $F(x' \bigtriangleup_i^{\lambda^*} y) = F(x')$. Thus $x' \bigtriangledown_i^{\lambda^*} y$ is in $\arg\max_{x\in S'} F(x)$ and $x' \bigtriangleup_i^{\lambda^*} y$ is in $\arg\max_{x\in S} F(x)$.

We prove the necessity part of the theorem by contradiction. Let x and y be elements in X where (2) is violated for $(\bigtriangledown, \bigtriangleup) = (\bigtriangledown_i^{\lambda^*}, \bigtriangleup_i^{\lambda^*})$ with λ^* in $[0, 1]$. This can happen only if $x_i > y_i$, and x' and y are not ordered. The elements x and $x \bigtriangleup_i^{\lambda^*} y$ must be distinct and, similarly, $x \bigtriangledown_i^{\lambda^*} y$ and y are distinct. Let $K = \{x,\ x \bigtriangleup_i^{\lambda^*} y\}$ and $K' = \{x \bigtriangledown_i^{\lambda^*} y,\ y\}$. Clearly, $K' \geq_i K$.

There are two possible violations of $\mathcal{C}_i$-quasisupermodularity. One possibility is that $F(x) \geq F(x \bigtriangleup_i^{\lambda^*} y)$, but $F(x \bigtriangledown_i^{\lambda^*} y) < F(y)$. Then x maximizes F in K and y *uniquely* maximizes F in K'. This violates the i-increasing property because $x_i > y_i$. The other possible violation of $\mathcal{C}_i$-quasisupermodularity is that $F(x) > F(x \bigtriangleup_i^{\lambda^*} y)$ but $F(x \bigtriangledown_i^{\lambda^*} y) = F(y)$. In this case, y maximizes F in K' while x is the unique maximizer of F in K. Again this violates the i-increasing property. *Q.E.D.*

The next result follows immediately from Theorem 2 and Proposition 4(i).

[16]Recall the definition of higher and i-higher sets in Section 2.

COROLLARY 1: *Let* $F : X \to R$ *be a* $\mathcal{C}_i$*-quasisupermodular function and let* $G : X \to R$ *be continuous, increasing, and* $\mathcal{C}_i$*-quasisubmodular. Then it holds that whenever* $k'' \geq k'$, *we have*

$$\operatorname*{arg\,max}_{x \in G^{-1}((-\infty, k''])} F(x) \geq_i \operatorname*{arg\,max}_{x \in G^{-1}((-\infty, k'])} F(x).$$

(In this case, we shall say that *the optimal value of* i *increases with* k, but bear in mind that we are not claiming that the optimal solutions are unique.)

Note that, by Proposition 2, we can modify the assumptions of Corollary 1: instead of $\mathcal{C}_i$-quasisupermodularity, we can assume that F is supermodular and i-concave, while we can replace the $\mathcal{C}_i$-quasisubmodularity of G by submodularity and i-convexity. These observations, together with Proposition 3, give us the next corollary.

COROLLARY 2: (i) *Let* $F : X \to R$ *be a supermodular and i-concave function, and let* $G : X \to R$ *be continuous, increasing, submodular, and i-convex. Then* $\arg\max_{x \in G^{-1}((-\infty, k''])} F(x)$ *is i-higher than* $\arg\max_{x \in G^{-1}((-\infty, k'])} F(x)$ *whenever* $k'' > k'$.

(ii) *Let* $F : X \to R$ *be a supermodular and partially concave function, and let* $G : X \to R$ *be continuous, increasing, submodular, and partially convex. Then* $\arg\max_{x \in G^{-1}((-\infty, k''])} F(x)$ *is higher than* $\arg\max_{x \in G^{-1}((-\infty, k'])} F(x)$ *whenever* $k'' > k'$.

Proposition 1 is clearly just a special case of Corollary 2 because, as we have already pointed out at the end of Section 4.2, the budget set $B(p, w)$ is the preimage of the continuous, increasing, convex, and submodular function $E(x) = p \cdot x$. So we have established Proposition 1.

4.4. *Related Literature and Other Issues*

We end this section with a discussion of the related literature and other issues. An early version of Proposition 1(ii) can be found in Chipman (1977). Chipman showed that if $U : R^l_{++} \to R$ is C^2, locally nonsatiated, differentiably strictly concave (i.e., U has a *strictly* negative-definite Hessian), and supermodular, then U generates a differentiable demand function that has normal demand for all goods, i.e., $\partial D_i(p, w)/\partial w \geq 0$ for all goods i. Chipman's proof uses properties of dominant diagonal matrices rather than the lattice programming-inspired methods of this paper, and relies on the smoothness and stronger concavity assumptions on U. The former assumption excludes the case of Choquet expected utility and other nonsmooth preferences, and the latter will exclude the case of constant and increasing returns, both of which are permitted in our approach (see Section 2). The more general case of changes to nonlinear constraint sets was not studied in his paper. Antoniadou (1995, 2004) and Mirman and Ruble (2003) have also investigated conditions for the normality of demand in the context of linear budget constraints. Their strategy is to find a lattice structure on the Euclidean space (other than that given by the product order) that will allow them to apply the standard comparative statics theorems for type B problems. The precise relationships among these different conditions is not immediately clear and is worth exploring.

A related question is the necessity of the properties imposed on U in Proposition 1. If we wish to obtain (as Proposition 1 does) only that increases in income lead to increases in demand, we could not appeal to Theorem 2 to conclude that $\mathcal{C}$-supermodularity is *necessary*: this is because such constraint set changes are just some—and not all—of that required in the necessity part of Theorem 2. We know that in the two goods case, if U is increasing, the $\mathcal{C}_2$-quasisupermodularity of U is equivalent to the indifference curves through (x_1, x_2) becoming flatter as x_1 increases (keeping x_2 fixed). It is very obvious that when U is quasiconcave, flattening indifference curves are also necessary for the normality of good 2. For the general l good case, it is not clear if $\mathcal{C}_i$-quasisupermodularity is necessary for the normality of good i when *only* budget sets with linear boundaries are permitted.

A well known and almost immediate corollary of Theorem 1 is that if the constraint set S is a sublattice of X and if F is supermodular, then $\arg\max_{x\in S} F(x)$ is also a sublattice. Indeed, in many contexts it is possible to show that this set is a subcomplete sublattice, so that one could sensibly speak of the smallest or largest element of $\arg\max_{x\in S} F(x)$ (see Topkis (1998, Corollary 2.7.1)). This is a useful feature that, among other things, facilitates the construction of an increasing selection. The constraint sets we consider in this paper are not, as a rule, sublattices (with respect to the product order, or indeed any lattice order), so one has to find some other way to construct increasing selections. Results on this problem can be found in Quah (2006).

Finally, a natural question that Proposition 2 raises is whether a $\mathcal{C}$-supermodular function must be partially concave (that it is supermodular is obvious). Subject to a continuity assumption on the function, the answer to this question is "Yes." An analogous question is whether $\mathcal{C}$-quasisupermodular functions are quasiconcave. It turns out that this is true provided the function is increasing. These issues are dealt with in Appendix B.

5. MORE EXAMPLES

In this section we present a number of examples that, in addition to being interesting applications in themselves, are also useful to illustrate various aspects of the theory developed in the last section.

EXAMPLE 3: The constraint set change considered in the two examples in Section 2 were of the relatively simple sort, involving changes to budget sets with linear boundaries. In this application the constraint set change is more complicated, but as we shall see, one can still identify reasonable conditions that guarantee comparability in the flexible set order. As in Example 2, we consider a firm that produces a single product using l inputs, with a revenue function V and production function F. We wish to examine the impact of a technological change on the optimal output: specifically, if $q = AF(x)$, how would an increase in A affect the optimal q?

It is convenient in this context to think of inputs as negative variables, so we define $\tilde{F}: R^l_- \to R$ by $\tilde{F}(\tilde{x}) = F(-\tilde{x})$. We can then formulate the firm's problem as a constrained maximization problem. Assuming that $p > 0$ is the input price vector, the firm's problem is to maximize $\Pi(\tilde{x}, q) = V(q) + p \cdot \tilde{x}$ subject to $(\tilde{x}, q)$ in $B_A = \{(x, q) \in R^l_- \times R_+ : q \leq A\tilde{F}(\tilde{x})\}$. To answer our question by applying Theorem 2, we require that Π be $\mathcal{C}_q$-quasisupermodular. Proposition 2 guarantees that this holds: the function Π is additive, and thus supermodular; it is also q-concave because it is linear in $\tilde{x}$. (It is also worth noting that because we have made no assumptions on V, Π is not generally partially concave: so here is an instance where the somewhat nuanced concavity properties we used in the last section are crucial in applications.)

Suppose that A increases from A' to A''. Then Theorem 2 guarantees an increase in optimal output if we have $B_{A''} \geq_q B_{A'}$. The next proposition gives the conditions required for this to hold and is proved in Appendix A.

PROPOSITION 5: *Suppose that F is continuous, supermodular, concave, and increasing. Then $B_{A''} \geq_q B_{A'}$ and* $\arg\max_{B_{A''}} \Pi(\tilde{x}, q) \geq_q \arg\max_{B_{A'}} \Pi(\tilde{x}, q)$ *if $A'' \geq A'$.*[17]

So we have found conditions on F that guarantee that optimal output increases with A. Note that the conclusion requires no assumptions on V. If the firm is a price-taker, with the price of the output normalized at 1, then it maximizes $\pi(x) \equiv AF(x) - p \cdot x$. The function π has increasing differences in $(x; A)$ provided F is an increasing function; thus an increase in A will increase the optimal x and hence output (which is $AF(x)$), provided F is supermodular. However, it is possible for output to increase even when optimal input falls. If the firm is not a price-taker, it maximizes $\pi(x) \equiv V(AF(x)) - p \cdot x$, so the optimal x decreases with A if the map from $(x; A)$ to $V(AF(x))$ has decreasing differences in $(x; A)$ and is supermodular in x. Notwithstanding this, output can still increase, as indeed it will if F obeys the conditions in Proposition 5.

The fact that the conclusion requires no assumptions on the revenue function V has to do with the fact that $\Pi(\tilde{x}, q) = V(q) + p \cdot \tilde{x}$ is $\mathcal{C}_q$-supermodular for any function V. This feature makes the function Π a trivial illustration of a general property: if a function f is $\mathcal{C}_l$-supermodular, then subjecting the

[17]For the interpretation of supermodularity, concavity, and other related conditions on the production function F, see the discussion in Example 2 and footnote 7.

variable x_l to an increasing transformation preserves $\mathcal{C}_l$-supermodularity. The next result states this formally; the proof is obvious so we shall skip it. We shall exploit this observation in the next example we present.

PROPOSITION 6: *Suppose that $X = \times_{i=1}^{l} X_i$, where each X_i is an interval on the real line. Let $\phi : \tilde{X}_l \to X_l$ be a strictly increasing function from the interval $\tilde{X}_l$ into X_l and define $\tilde{X} = \times_{i=1}^{l-1} X_i \times \tilde{X}_l$. Then the following conditions hold*:

(i) *If the function $f : X \to R$ is $\mathcal{C}_l$-supermodular ($\mathcal{C}_l$-quasisupermodular), then so is $\tilde{f} : \tilde{X} \to R$, where $\tilde{f}(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_l) = f(\tilde{x}_1, \tilde{x}_2, \dots, \phi(\tilde{x}_l))$.*

(ii) *If $K'' \geq_l K'$ for K'' and K' in X, then $\tilde{K}'' \geq_l \tilde{K}'$ for $\tilde{K}''$ and $\tilde{K}'$ in $\tilde{X}$, where $\tilde{K}'' = \{\tilde{x} \in \tilde{X} : (\tilde{x}_1, \tilde{x}_2, \dots, \phi(\tilde{x}_l)) \in K''\}$ and $\tilde{K}'$ is defined analogously.*

EXAMPLE 4: In this example, we show how the theory developed in this paper casts a new light on Rybcsynski's theorem and identifies ways in which the theorem can be extended.[18] The context of the theorem is an open economy that produces goods A and B using two factors: capital and labor. There is no joint production; good i ($i = A, B$) has the production function $f_i : R^2_+ \to R_+$, with f_i quasiconcave and homogeneous of degree 1 (in other words, has constant returns to scale). These two assumptions also guarantee that f_i is concave (see Champsaur and Milleron (1983)). The economy has an endowment $\bar{K}$ of capital and $\bar{L}$ of labor. Production decisions in this economy are made by two representative firms: firm i chooses (k_i, l_i) in R^2_+ to maximize $\Pi_i(l_i, k_i) = V_i(f_i(k_i, l_i)) - w_K k_i - w_L l_i$, where $V_i(x_i)$ is the revenue earned from selling x_i units of good i, and w_K and w_L are prices of capital and labor, respectively. An equilibrium in this economy is reached when w_K and w_L are such that the firms' demand for capital and labor equals the economy's endowments. It is well known that under standard assumptions, the equilibrium output of goods A and B can also be obtained via an optimization procedure. Let $S(\bar{K}, \bar{L})$ in R^2_+ be the production possibility set of this economy when the aggregate endowment is $(\bar{K}, \bar{L})$. Then the equilibrium output of the two goods coincides with that obtained from the following optimization problem: maximize $U(a, b) = V_A(a) + V_B(b)$ subject to (a, b) in $S(\bar{K}, \bar{L})$.

Suppose that the markets for the two goods are perfectly competitive, so $V_A(a) = p_A a$ and $V_B(b) = p_B b$. Rybcsynski's theorem considers an increase in the endowment of capital, say from $\bar{K}'$ to $\bar{K}''$, and identifies conditions on the relationship between f_A and f_B that guarantee that *more of good B and less of good A* is produced at equilibrium. (See Mas-Colell, Whinston, and Green (1995); for reasons of brevity, we will not examine those conditions.) This is illustrated in Figure 2, where the optimal output moves from (a^*, b^*) to

[18]For a generalization of another classic result in trade theory—the Stolper–Samuelson theorem—using the methods of monotone comparative statics, see Echenique and Manelli (2003).

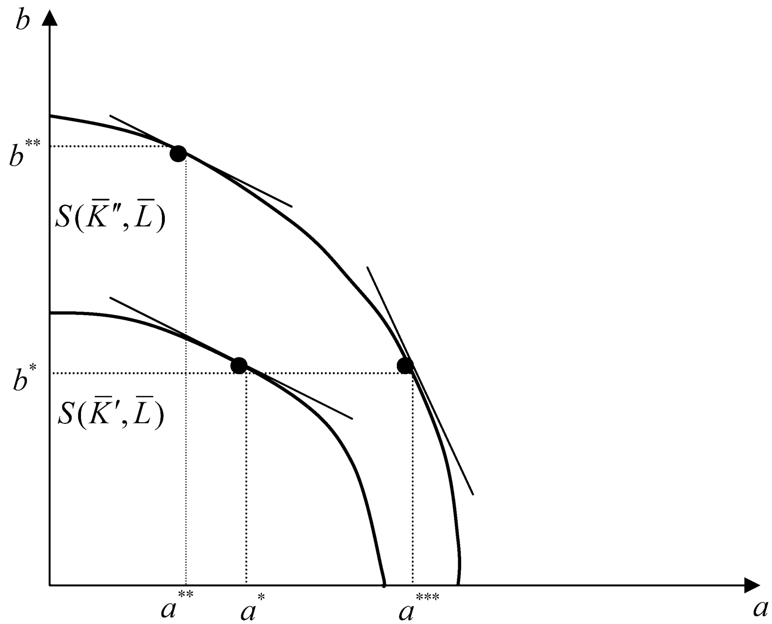

FIGURE 2.

(a^{**}, b^{**}); because output prices are held fixed, the slope of the production possibility frontiers at these points are the same. Note that because the production functions are concave, the production possibility sets are convex, so the slope of the new production possibility set at (a^{***}, b^{*}) must be steeper than the slope at (a^{*}, b^{*}).[19] This means that, following from the increase in capital, we *must* have $S(\bar{K}'', \bar{L}) \geq_b S(\bar{K}', \bar{L})$. (This is intuitive; see Appendix A for the proof.) We have a situation where Theorem 2 is applicable. By Theorem 2 and Proposition 2, the optimal level of B will rise whenever R_A is concave (so that U is $\mathcal{C}_2$-supermodular) and does not require either V_A or V_B to be linear. However, with this departure, we will not be able to say that the output of good A falls.

We now show that the increase in the optimal output of B does not depend on B's production function having constant returns to scale. To see this, consider an increasing transformation of the production function of good B from f_B to $\hat{f}_B = H \circ f_B$, where $H : R_+ \to R_+$ is a strictly increasing function. Like f_B, $\hat{f}_B$ will be quasiconcave, but it will not, in general, be concave or have constant returns to scale. We denote the production possibility sets generated by f_A and $\hat{f}_B$ as $\hat{S}$; because $\hat{f}_B$ need not be concave, neither must $\hat{S}$. One can

[19]It is not hard to prove this claim directly from the assumptions in Rybcsynski's theorem, if one so wishes.

easily check that $\hat{S}(\bar{K}', \bar{L}) = \{(a, b) \in R^2_+ : (a, H^{-1}(b)) \in S(\bar{K}', \bar{L})\}$; obviously, $\hat{S}(\bar{K}'', \bar{L})$ is similarly related to $S(\bar{K}'', \bar{L})$. Given that $S(\bar{K}'', \bar{L}) \geq_b S(\bar{K}', \bar{L})$, Proposition 6(ii) guarantees that we must also have $\hat{S}(\bar{K}'', \bar{L}) \geq_b \hat{S}(\bar{K}', \bar{L})$. It follows that the output of B will rise with the endowment of capital: the comparative statics result is robust to increasing transformations of the production function of good B.

EXAMPLE 5: In all the examples considered so far, obtaining the $\mathcal{C}$-quasisupermodularity of the objective function involved a straightforward application of Proposition 2. This is not always so. Sometimes deriving this property from assumptions natural to a particular context requires us to apply Proposition 2 in a less obvious way.

We consider the following two period saving-portfolio problem. At date 1, an agent decides how much to save out of date 1 income w_1. Savings can be invested in two ways: a risk-free asset A, which pays off $r > 0$ at date 2 and an asset B that has a stochastic payoff of s at date 2, with s distributed according to the density function f. In addition, the agent has a nonstochastic income of w_2 at date 2. The agent's Bernoulli utility function is $u(c_1, c_2)$, where c_1 and c_2 refer to consumption at dates 1 and 2, respectively. If he holds a portfolio with a of asset A and b of asset B, his utility given a date 1 consumption of c_1 is $U(c_1, a, b) = \int u(c_1, ar + bs + w_2) f(s)\, ds$. Without loss of generality, we assume that both assets are priced at 1. The agent then solves the following problem **P**: maximize $U(c_1, a, b)$ subject to $c_1 + a + b \leq w_1$. We assume that **P** obeys the *regularity conditions* that u is C^2 and concave, $u_2(c_1, c_2) > 0$ for all (c_1, c_2), **P** has a unique solution that obeys by the first-order conditions, and varies smoothly with w_1.

A fundamental question we can ask is how the agent's savings and investments will vary with w_1.[20] It is common practice in this context to assume that u is additively separable, i.e., $u(c_1, c_2) \equiv v(c_1) + \bar{v}(c_2)$, with v and $\bar{v}$ concave. With this assumption, the answer to our question is an easy corollary of known results. For any given amount saved, the agent chooses a portfolio comprised of A and B to maximize the expected value of $\bar{v}$. Standard portfolio theory tells us that, provided $\bar{v}$ has decreasing absolute risk aversion, investment in the risky asset B increases with savings. The agent decides on the amount saved in period 1 by maximizing the sum of period 1 utility and the expected utility of period 2 consumption. The latter is a concave function of savings provided the portfolio is chosen optimally. It is straightforward to show that in such a situation, an increase in w_1 increases *both* savings and consumption. Thus we obtain that *an increase in* w_1 *causes period* 1 *consumption, savings, and investment in the risky asset to rise.* Our objective is to reach the same conclusion in the case where u is not additively separable.

[20] For a discussion of other interesting issues in the saving-portfolio problem, see Gollier (2001).

Problem **P** is a demand problem much like Example 1, so a natural impulse would be to apply Proposition 1, but a straightforward application to this problem is not possible. Observe that $U_{ab}(c_1, a, b) = \int u_{22}(c_1, ar + bs + w_2)rsf(s)\,ds$. Given risk aversion, u_{22} will always be negative; if in addition, rs is positive, then U_{ab} will be negative, so U is not a supermodular function. However, we can reformulate the agent's problem by imagining him choosing between two assets: a riskless asset A, priced at 1 and with a constant payoff r, and a risky asset X, priced at zero, which has payoff $t = s - r$. Asset X can be constructed by buying one B and selling one A. Formally, the agent solves $\mathbf{P}'$: maximize $\tilde{U}(\tilde{c}_1, \tilde{a}, x) = \int u(\tilde{c}_1, \tilde{a}r + xt + w_2)g(t)\,dt$ subject to $\tilde{c}_1 + p\tilde{a} = w_1$, where g is the density function of t. Suppose that at w_1, the solution to **P** is $(c_1^*(w_1), a^*(w_1), b^*(w_1))$ and the solution to $\mathbf{P}'$ is $(\tilde{c}_1^*(w_1), \tilde{a}^*(w_1), x^*(w_1))$. These solutions are related in the manner $c_1^*(w_1) = \tilde{c}_1^*(w_1)$, $b^*(w_1) = x^*(w_1)$, and $a^*(w_1) = \tilde{a}^*(w_1) - x^*(w_1)$. Because $u_2 > 0$, the budget constraint must bind at the optimum, so that savings is $w_1 - c_1^*(w_1) = a^*(w_1) + b^*(w_1) = \tilde{a}^*(w_1)$. This means that if $(\tilde{c}_1^*(w_1), \tilde{a}^*(w_1), x^*(w_1))$ increases with w_1, we obtain precisely the conclusion we want: period 1 consumption, savings, and investment in the risky asset B all increase with w_1.[21]

Because we have assumed that the solution to **P** varies continuously with w_1, so does the solution to $\mathbf{P}'$. Therefore, Proposition 1 guarantees that $(\tilde{c}_1^*(w_1), \tilde{a}^*(w_1), x^*(w_1))$ increases with w_1 if $\tilde{U}$ is partially concave and supermodular in some neighborhood of the optimum.[22] The first property is immediate because u is concave, so it remains for us to show the local supermodularity of $\tilde{U}$. The following proposition is proved in Appendix A.

PROPOSITION 7: *Suppose that the problem* **P** *obeys the regularity conditions and let* $(\tilde{c}_1^*(w_1), \tilde{a}^*(w_1), x^*(w_1))$ *be the solution to* $\mathbf{P}'$ *at* w_1. *Then* $\tilde{U}$ *is locally supermodular at* $(\tilde{c}_1^*(w_1), \tilde{a}^*(w_1), x^*(w_1))$ *if the following conditions hold*:

(a) *For all* c_1, $-u_{22}(c_1, c_2)/u_2(c_1, c_2)$ *strictly decreases with* c_2, *i.e., there is strictly decreasing risk aversion*.

(b) *For all* (c_1, c_2), $u_{12}(c_1, c_2) > 0$.

(c) *For all* c_1, $u_{12}(c_1, c_2)/u_2(c_1, c_2)$ *strictly increases with* c_2.

Let $v : R_{++} \to R$ be a function with $v'(z) > 0$, $-v''(z)/v'(z)$ strictly decreasing in z, and $-zv''(z)/v'(z)$ strictly increasing in z. Then it is not hard to check that $u(c_1, c_2) = c_1 v(c_2/c_1)$ is an example of a function that satisfies (a), (b), and (c) in Proposition 7. Condition (a) in the proposition is familiar and should be

[21]Because we do not know the relative magnitudes of the changes to $x^*(w_1)$ and $\tilde{a}^*(w_1)$ given an increase in w_1, we cannot determine if the agent will buy more or less of A.

[22]More carefully, because the optimal portfolio varies continuously with w_1, any violation of normality must mean a local violation of normality, but this is impossible because there is always an open neighborhood around each optimum in which Proposition 1 is applicable.

expected given what we know of the problem when u is time separable. Condition (b)—the supermodularity of u—is usually interpreted as habit formation and has been extensively considered as an explanation of the equity premium puzzle (see, for example, Constantinides (1990)). Condition (c) is less familiar but has a simple interpretation, which we now give.

The fact that $u_{12} > 0$ means that for any $\Delta > 0$,

$$\frac{1}{2}u(c_1 + \Delta, c_2 + \Delta) + \frac{1}{2}u(c_1, c_2) \geq \frac{1}{2}u(c_1, c_2 + \Delta) + \frac{1}{2}u(c_1 + \Delta, c_2).$$

So the complementarity of c_1 and c_2 can be understood to mean that the agent will always prefer a 50–50 gamble between the bundles $(c_1 + \Delta, c_2 + \Delta)$ and (c_1, c_2) to a 50–50 gamble between the bundles $(c_1, c_2 + \Delta)$ and $u(c_1 + \Delta, c_2)$. As long as u is increasing, the agent can be given a bit more consumption at date 2 (a "premium") in the second gamble such that she would be indifferent between these gambles. Formally, there is $\pi > 0$ (π is a function of c_1, c_2, and Δ) such that

$$u(c_1 + \Delta, c_2 + \Delta) + u(c_1, c_2) \equiv u(c_1, c_2 + \Delta + \pi) + u(c_1 + \Delta, c_2 + \pi).$$

By Taylor's expansion around (c_1, c_2), one can check that

$$\lim_{\Delta \to 0} \frac{\pi(c_1, c_2, \Delta) - \frac{u_{12}(c_1, c_2)}{u_2(c_1, c_2)} \frac{\Delta^2}{2}}{\Delta^2} = 0.$$

In other words, when Δ is small, π is approximately $(u_{12}/u_2)(\Delta^2/2)$.[23] Condition (c) means that for $c_2'' > c_2'$, we have $\pi(c_1, c_2'', \Delta) > \pi(c_1, c_2', \Delta)$ when Δ is sufficiently small.

EXAMPLE 6: This paper has focussed on type B comparative statics problems, but some problems have both type A and type B features and may be usefully addressed by combining the results in this paper with results for type A problems. We illustrate this with a study of the short and long run response of a firm to a change in input price.

The classic formulation of the LeChatelier principle in economics considers the impact of a *small* reduction in the price of an input (say input 1) on the demand for 1. It says that in the short run, interpreted as the time frame in which some inputs are not free to vary, the increase in the demand for 1 is smaller than in the long run, when all inputs are free to vary. Milgrom and Roberts (1996) have shown that this result also holds even when the price reduction is large, provided the profit function is a supermodular function of the inputs. Our next result gives a formulation of the LeChatelier principle that

[23]Notice that our interpretation of u_{12}/u_2 is analogous to the standard interpretation given to the coefficient of absolute risk aversion (see Mas-Colell, Whinston, and Green (1995)).

extends the result of Milgrom and Roberts by enlarging the class of permissible constraints faced by the firm in the short run.[24] We adopt the notation we use in Example 3, so V represents the revenue from output, F is the production function, and x is the input vector. The input price of 1 is $(p_1 - a)$; that of good i, for $i \geq 2$, is given by p_i. Hence the firm's profit function may be written as $\Pi(\cdot, a): R^l_+ \to R$, where $\Pi(x, a) = V(F(x)) - p \cdot x + ax_1$. (Note that the vector $p = (p_1, p_2, \dots, p_l)$.)

PROPOSITION 8: *Let x^* be a solution to the problem* (i) *maximize $\Pi(x, a')$ subject to x in R^l_+. Suppose, also, that there are solutions to the problems* (ii) *maximize $\Pi(x, a'')$ subject to $x \in C$, where $a'' > a'$ and C is a subset of R^l_+ that contains x^*, and* (iii) *maximize $\Pi(x, a'')$ subject to x in R^l_+. Then there are x^{**} and x^{***}, solutions to problems* (ii) *and* (iii), *respectively, such that $x_1^{***} \geq x_1^{**} \geq x_1^*$, provided* either *of the following conditions holds*:

(A) *The function $V \circ F$ is supermodular and X_C is greater than C in the strong set order, where $X_C = \{x \in R^l_+ : x \geq c$ for some $c \in C\}$.*

(B) *The function $V \circ F$ is supermodular and* 1*-concave, and X_C is greater than C in the $\mathcal{C}$-flexible set order.*

The proof of this result is in Appendix A. The proposition gives the direction of change in the firm's optimal input of 1 in the short and long run following a fall in the price of input 1 from $p_1 - a'$ to $p_1 - a''$. The short run comparative statics of the firm (case (ii)) is a type A problem, where the constraint set remains unchanged at C, but the firm's objective changes because an input price has changed. The comparative statics between short and long run (case (iii)) can be understood as a type B problem because it can be formulated as a change in the firm's constraint set from C to R^l_+.

The desired conclusion holds under two sets of assumptions. In both cases, $V \circ F$ is assumed to be supermodular, which guarantees the supermodularity of the profit function Π. In (B), $V \circ F$ (and thus Π) is also assumed to be 1-concave, but the class of constraint sets permitted is larger than in (A), because the set ordering requirements is weaker.

Milgrom and Roberts (1996) considered the case where certain inputs are held fixed in the short run. Formally, they are considering a constraint set of the form $C = \{x \in R^l_+ : x_i = k_i$ for $i = m, m+1, \dots, l\}$. It is quite obvious that in this case X_C is greater than C in the strong set order, so that condition (A) may be applied, but this is just one of many possible types of constraints that a firm might face in the short run. It is not hard to check that X_C dominates C in the strong set order for any set C that has the *free disposal* property that if x' is in C, then $x > 0$ with $x < x'$ is also in C. Short run constraints of this form

[24]For a discussion of the LeChatelier principle, including an extension in a direction different from the one considered here (or in Milgrom and Roberts (1996)), see Roberts (1999).

are quite plausible; for example, a firm that in the short run cannot allow its expenditure to exceed w will formally have $C = \{x \in R^l_+ : p \cdot x - a'' x_1 \leq w\}$.

For an example of the type of short run constraint permitted by condition (B), let $\phi : R^l_+ \to R$ be a continuous and increasing function and let r be in the range of ϕ. The set $C = \phi^{-1}(r)$ is nonempty and closed, and X_C dominates C in the $\mathcal{C}$-flexible set order.[25] It is also easy to see that X_C need not dominate C in the strong set order (thus (A) is inapplicable). Short run constraints of this form are economically plausible. For example, suppose that the inputs $m, m+1, \ldots, l$ are intrinsically the same good, i.e., they have the same inherent characteristics, but are only considered as different inputs because they play different roles in the production process. Imagine that in the short run, due to contractual or technological reasons, the total amount of this good used cannot be varied, although the firm is free to employ what they already have in different ways. In that case, $C = \{x \in R^l_+ : \sum_{i=m}^l x_i = \sum_{i=m}^l x_i^*\}$, which equals $\phi^{-1}(r)$ if we define $\phi(x) = \sum_{i=m}^l x_i$ and $r = \sum_{i=m}^l x_i^*$.

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APPENDIX A

PROOF OF THEOREM 1: Both parts can be proven by a superficial modification of the proof of Milgrom and Shannon (1994, Theorem 4). We shall only prove the "only if" part. Suppose x' is in $\arg\max_{x \in S} f(x)$ and y is in $\arg\max_{x \in S'} f(x)$. Given that $S' \geq_{(\triangledown, \triangle)} S$, $x' \triangledown y$ is in S' and $x' \triangle y$ is in S. By revealed preference, $f(x') \geq f(x' \triangle y)$; the $(\triangledown, \triangle)$-quasisupermodularity of f then guarantees that $f(x' \triangledown y) \geq f(y)$, which implies that $x' \triangledown y$ is in $\arg\max_{x \in S'} f(x)$. Now suppose that $x' \triangle y$ is not in $\arg\max_{x \in S} f(x)$ and thus $f(x') > f(x' \triangle y)$. Because f is $(\triangledown, \triangle)$-quasisupermodular, we obtain $f(x' \triangledown y) > f(y)$, contradicting the assumption that y is in $\arg\max_{x \in S'} f(x)$. *Q.E.D.*

PROOF OF PROPOSITION 5: The second claim follows from Theorem 2 if $B_{A''} \geq_q B_{A'}$. Define the function G by $G(\tilde{x}, q) = q/A' - \tilde{F}(\tilde{x})$ and consider $x' = (\tilde{x}', q')$ in $B_{A'} = G^{-1}((-\infty, 0])$ and $x'' = (\tilde{x}'', q'')$ in $B_{A''}$ with $q' > q''$. Assume that x'' is not in $B_{A'}$, so $G(x'') > 0$, the other case being trivial. Note that G is continuous, increasing, and $\mathcal{C}_q$-submodular. By Proposition 4, $G^{-1}((-\infty, G(x'')]) \geq_q G^{-1}((-\infty, 0]) = B_{A'}$, so there is λ in $[0, 1]$ such that $x' \triangle_q^\lambda x''$ is in $B_{A'}$ and $z = x' \triangledown_q^\lambda x''$ with $G(z) \leq G(x'')$. We claim that z is

[25] Let x be in C and let y be in X_C with $x_i > y_i$. Then $\phi(y) > r$, while $\phi(x \wedge y) \leq r$ (since ϕ is increasing). Thus there is λ in $[0, 1]$ such that $\phi(x \triangle_i^\lambda y) = r$. Clearly $x \triangle_i^\lambda y$ is in $\phi^{-1}(r) = C$, while $x \triangledown_i^\lambda y$ is in X_C.

in $B_{A''}$. Because x'' is in $B_{A''}$, $G(x'') + (q''/A'') - (q''/A') \le 0$, so $G(z) + (q''/A'') - (q''/A') \le 0$. Note that the last entry of the vector z is q' and because the map from (q, A) to q/A is submodular, we have $(q'/A'') - (q'/A') < (q''/A'') - (q''/A')$. Thus $G(z) + (q'/A'') - (q'/A') \le 0$, which implies that z is in $B_{A''}$. *Q.E.D.*

In Example 4, we claimed that $S(\bar{K}'', \bar{L}) \ge_b S(\bar{K}', \bar{L})$ because whenever we compare points (a^{***}, b^*) and (a^*, b^*), the boundary of $S(\bar{K}'', \bar{L})$ at (a^{***}, b^*) is steeper than the boundary of $S(\bar{K}', \bar{L})$ at (a^*, b^*) (see Figure 2). We now state and prove this formally.

LEMMA A1: *Suppose that the boundary of $S(\bar{K}', \bar{L})$ is given by the decreasing and differentiable function $g:(0, B^*) \to R_+$, and the boundary of $S(\bar{K}'', \bar{L})$ is given by the decreasing and differentiable function $h:(0, B^{**}) \to R_+$, with the properties $B^{**} \ge B^*$, $h(b) \ge g(b)$, and $h'(b) \ge g'(b)$ for all $b \le B^*$. Then $S(\bar{K}'', \bar{L}) \ge_b S(\bar{K}', \bar{L})$.*

PROOF: Suppose (a', b') is in $S(\bar{K}', \bar{L})$ and (a'', b'') is in $S(\bar{K}'', \bar{L})$ but not in $S(\bar{K}', \bar{L})$, with $b' > b''$ and $a'' > a'$. Note that $(b'', g(b''))$ is in $S(\bar{K}', \bar{L})$ and $g(b'') \le g(b')$ because g is decreasing. We wish to show that the point $(a'' - (g(b'') - a'), b')$ is in $S(\bar{K}'', \bar{L})$. Given that $g(b') \ge a'$, it is sufficient to show that $(a'' - (g(b'') - g(b')), b')$ is in $S(\bar{K}'', \bar{L})$; this is true if $h(b') \ge a'' - (g(b'') - g(b'))$. Rearranging this expression gives

$$g(b') - g(b'') \le h(b') - a''.$$

The assumption that $g'(b) \le h'(b)$ guarantees that $g(b') - g(b'') \le h(b') - h(b'')$; furthermore, $h(b') - h(b'') \le h(b') - a''$ because $a'' \le h(b'')$. *Q.E.D.*

PROOF OF PROPOSITION 7: Differentiating $\tilde{U}$ by $\tilde{a}$ and x, we obtain

$$\begin{aligned}\tilde{U}_{\tilde{a},x}(\tilde{c}_1^*, \tilde{a}^*, x^*) &= \int u_{22}(\tilde{c}_1^*, \tilde{a}^* r + x^* t + w_2) r t g(t)\, dt \\ &= r \int \left[-\frac{u_{22}}{u_2}\right](-u_2(\tilde{c}_1^*, \tilde{a}^* r + x^* t + w_2)) t g(t)\, dt.\end{aligned}$$

Let k_0 be the coefficient of risk aversion at $(\tilde{c}_1^*, \tilde{a}^* r + w_2)$ (when $t = 0$). Since the coefficient is strictly decreasing,

$$\tilde{U}_{\tilde{a},x}(\tilde{c}_1^*, \tilde{a}^*, x^*) > r k_0 \int (-u_2(\tilde{c}_1^*, \tilde{a}^* r + x^* t + w_2)) t g(t)\, dt = 0,$$

where the last equality follows from the first order condition $\tilde{U}_x(\tilde{c}_1^*, \tilde{a}^*, x^*) = 0$ (recall that the price of X is zero). Because $u_{12} > 0$ by assumption,

$$\tilde{U}_{\tilde{c}_1,\tilde{a}}(\tilde{c}_1^*, \tilde{a}^*, x^*) = \int u_{12}(c_1^*, \tilde{a}^* r + x^* t + w_2) r g(t)\, dt > 0$$

in some neighborhood of the optimum. Finally,

$$\tilde{U}_{\tilde{c}_1,x}(c_1^*, \tilde{a}^*, x^*) = \int u_{12}(\tilde{c}_1^*, \tilde{a}^* r + x^* t + w_2) t g(t)\, dt$$
$$= \int \left[\frac{u_{12}}{u_2}\right](u_2(\tilde{c}_1^*, \tilde{a}^* r + x^* t + w_2)) t g(t)\, dt.$$

The last term can be shown to be positive by applying an argument similar to the one we used for $\tilde{U}_{\tilde{a},x}$. *Q.E.D.*

The proof of Proposition 8 requires the following simple comparative statics result for type A problems.

LEMMA A2: *Let T be a subset of R and let C be a subset of R^l. The function F maps $C \times T$ to R, with $F(x,t) = \bar{F}(x) + f(x_1, t)$, where $f : R \times T \to R$ is a supermodular function. Suppose that x' is in $\arg\max_{x \in C} F(x, t')$ and x'' is in $\arg\max_{x \in C} F(x, t'')$. If $x'_i > x''_i$, then x' is in $\arg\max_{x \in C} F(x, t'')$ and x'' is in $\arg\max_{x \in C} F(x, t')$. So, in particular, $\arg\max_{x \in C} F(x, t'') >_i \arg\max_{x \in C} F(x, t')$.*

PROOF[26]: Endow R^l with the lexicographic order, i.e., $x' > x$ if and only if there is i such that $x'_i > x_i$ and $x'_j = x_j$ for all $j < i$. This order is complete, so any function, $F(\cdot, t)$ in particular, is supermodular. Completeness also guarantees that any subset C is a sublattice. It is not hard to check that the supermodularity of f guarantees that the family $\{F(\cdot, t)\}_{t \in T}$ has the single crossing property in $(x; t)$. The result then follows from Milgrom and Shannon (1994, Theorem 4). *Q.E.D.*

PROOF OF PROPOSITION 8: Because x^* is in C and the map $(x_1, a) \to a x_1$ is supermodular, Lemma A2 guarantees that there is x^{**} solving problem (ii) such that $x_1^{**} \geq x_1^*$. (Note that this parameter change is more specific than the general parameter change considered by Milgrom and Roberts (1996).)

Given that in both (A) and (B), Π is supermodular in x and has increasing differences in (x, a), and that $\arg\max_{x \in R^l_+} \Pi(x, a'')$ is nonempty by assumption, there is some $\bar{x}$ in $\arg\max_{x \in R^l_+} \Pi(x, a'')$ such that $\bar{x} \geq x^*$. Because x^* is in C, $\bar{x}$ also solves $\max_{x \in X_C} \Pi(x, a'')$; in particular, $\arg\max_{x \in X_C} \Pi(x, a'')$ is nonempty. We also know that $\arg\max_{x \in C} \Pi(x, a'')$ contains x^{**}. Thus there is x^{***} in $\arg\max_{x \in X_C} \Pi(x, a'')$ such that $x_1^{***} \geq x_1^{**}$: in case (A) this follows from Theorem 1; in case (B) it follows from Theorem 2. Finally note that x^{***} is in $\arg\max_{x \in R^l_+} \Pi(x, a'')$ because $\bar{x}$ is in X_C and also in $\arg\max_{x \in R^l_+} \Pi(x, a'')$. *Q.E.D.*

[26] I am grateful to an anonymous referee for suggesting this proof.

APPENDIX B

In this appendix, we examine the implications that $\mathcal{C}$-supermodularity and $\mathcal{C}$-quasisupermodularity have on the concavity of a function. Let I be an interval in R. A function $g: I \to R$ is *concave* if $g(\alpha s + (1-\alpha)s') \geq \alpha g(s) + (1-\alpha)g(s')$ for all s and s' in I and α in $[0, 1]$. We say that the function g is **concave* (**convex*) if

$$g(s') - g(s) \geq (\leq)\, g(s'+c) - g(s+c) \tag{7}$$

whenever $s < s'$, $c > 0$, and the four points concerned lie in I. A function $f: X \to R$ is said to be *concave in direction $v \neq 0$ if for all x in X, the map from the scalar s to $f(x + sv)$ is *concave. The domain of this map is taken to be the largest possible interval such that $x + sv$ lies in X. We can then define i-*concave, i-*convex, partially *concave, and partially *convex functions, modified in the obvious way from our definition of i-concave functions, etc.

It is straightforward to check that if $g: I \to R$ is concave, then it is also *concave. *Assuming that g is continuous*, it is also not hard to check that a *concave function is concave. However, the continuity assumption cannot be dropped. It is known that there is a function $H: R \to R$ that obeys $H(s + s') = H(s) + H(s')$ and is discontinuous at all points on R (see Hardy, Littlewood, and Polya (1952)). Because any standard concave function must be continuous on the interior of its domain, H is not concave. On the other hand, H clearly satisfies *concavity.

If one examines the results in this paper where concavity or convexity is assumed, it will become clear that it is the weaker, starred property that is actually used in the proofs. In particular, in Proposition 2, the i-concavity and partial concavity conditions could be replaced by their starred versions. We shall now study the extent to which *concavity is implied by $\mathcal{C}$-supermodularity.

Our first result is a slightly different formulation of $\mathcal{C}$-supermodularity and quasisupermodularity that will be convenient for expositional purposes. We skip the obvious proof.

LEMMA B1: (i) *The function $f: X \to R$ is $\mathcal{C}_i$-supermodular if and only if for any vector v such that $v_i < 0$ and $v \not< 0$, we have*

$$f(x + \lambda v_+) - f(x + v + \lambda v_+) \geq f(x) - f(x + v),$$

where λ is a positive scalar and $v_+ = v \vee 0$.

(ii) *The function $f: X \to R$ is $\mathcal{C}_i$-quasisupermodular if and only if for any vector v such that $v_i < 0$ and $v \not< 0$, we have*

$$f(x) - f(x + v) \geq (>)\, 0 \Longrightarrow f(x + \lambda v_+) - f(x + v + \lambda v_+) \geq (>)\, 0,$$

where λ is a positive scalar and $v_+ = v \vee 0$.

The next result says that, upon adding a mild assumption, a function that is $\mathcal{C}_i$-supermodular is i-*concave.

PROPOSITION B1: *Let $X \subset R^l$ be a convex and open sublattice, and let $f : X \to R$ be a $\mathcal{C}_i$-supermodular function that is continuous in x_i. Then f is i-*concave.*

PROOF: Suppose, by way of contradiction, that there is $\bar{v} > 0$ with $\bar{v}_i = 0$ and $\lambda > 0$ such that $f(x) - f(x + \bar{v}) > f(x + \lambda\bar{v}) - f(x + \bar{v} + \lambda\bar{v})$. Because f is continuous in x_i and X is open, there is $\delta > 0$ and sufficiently close to zero such that $f(x) - f(x + \bar{v} - \delta e_i) > f(x + \lambda\bar{v}) - f(x + \bar{v} - \delta e_i + \lambda\bar{v})$, where e_i is the unit vector pointing in direction i. By Lemma B1, this is a violation of $\mathcal{C}_i$-supermodularity because $(\bar{v} - \delta e_i) \vee 0 = \bar{v}$. *Q.E.D.*

The next result says that $\mathcal{C}$-supermodular functions are *concave in all directions except the strictly positive and the strictly negative.

PROPOSITION B2: *Let $X \subset R^l$ be a convex and open sublattice, and suppose that $f : X \to R$ is a $\mathcal{C}$-supermodular function that is also continuous in each of its arguments (but not necessarily jointly continuous). Then f is *concave in all directions v that satisfy $v \not\gg 0$ and $v \not\ll 0$. In particular, f must be partially *concave.*

PROOF: For $v > 0$ (and, thus, also for $v < 0$) such that $v_i = 0$ for some i, we can appeal to Proposition B1. (Note that this is the only place where the continuity property imposed on f and the openness of X is used.) We now turn to the case where v is such that $v_i < 0$ for some i and $v_j > 0$ for some j. Let s be a positive scalar. Denote $v_+ = v \vee 0$ and $v_- = v \wedge 0$. We have

$$\begin{aligned} f(x) - f(x+v) &\le f(x + sv_+) - f(x + v + sv_+) \\ &\le f(x + sv_+ + sv_-) - f(x + v + sv_+ + sv_-) \\ &= f(x + sv) - f(x + v + sv). \end{aligned}$$

The first inequality arises from $\mathcal{C}_i$-supermodularity and the second arises from $\mathcal{C}_j$-supermodularity. (Note that all the elements referred to in the inequalities are in X because it is a convex lattice.) *Q.E.D.*

As a simple illustration, consider the function $f : R^2_{++} \to R$ given by $f(x_1, x_2) = x_1 x_2$. This function is continuous, partially concave, and supermodular. By Proposition 2, it is $\mathcal{C}$-supermodular, which means by Proposition B2 that it is concave in all directions except possibly those that are strictly positive or strictly negative. To check this, consider the behavior of the function along the ray emanating from the point $(\bar{x}_1, \bar{x}_2)$ and in the direction (a, b):

$f(\bar{x}_1 + as, \bar{x}_2 + bs) = \bar{x}_1\bar{x}_2 + (b\bar{x}_1 + a\bar{x}_2)s + abs^2$, which is a concave function of s whenever a and b are of different signs, but convex whenever a and b are *both* strictly positive or strictly negative.

The results we have reported so far have analogs for $\mathcal{C}$-quasisupermodular functions. A function $g: I \to R$ is *quasiconcave* if, for any M, the set $\{x \in I : g(x) \geq M\}$ is convex. The function is **quasiconcave* if for all $s < s'$ and $c > 0$, $g(s) > g(s')$ implies that $g(s + c) \geq g(s' + c)$. The latter property is implied by the former, and it is also not hard to check that the converse is true, provided g is continuous. Once again, the function H shows that continuity cannot be dropped for this result. This is because a quasiconcave function can have only a countable number of discontinuities, whereas H, which is *quasiconcave, is discontinuous everywhere.[27]

The next result is analogous to Proposition B1.

PROPOSITION B3: *Let $X \subset R^l$ be a convex and open sublattice and let $f: X \to R$ be a $\mathcal{C}_i$-quasisupermodular function that is continuous in x_i. Then f is i-*quasiconcave.*

PROOF: Suppose, by way of contradiction, that there is $\bar{v} > 0$ with $\bar{v}_i = 0$, and a positive scalar λ such that $f(x) - f(x + \bar{v}) > 0$ and $f(x + \lambda\bar{v}) - f(x + \bar{v} + \lambda\bar{v}) < 0$. Given that f is continuous in x_i and X is open, there is $\delta > 0$ and sufficiently close to zero such that $f(x) - f(x + \bar{v} - \delta e_i) > 0$ and $f(x + \lambda\bar{v}) - f(x + \bar{v} - \delta e_i + \lambda\bar{v}) < 0$, where e_i is the unit vector pointing in direction i. By Lemma B1, this is a violation of $\mathcal{C}_i$-quasisupermodularity because $(\bar{v} - \delta e_i) \wedge 0 = \bar{v}$. *Q.E.D.*

Our final result shows that any continuous and increasing $\mathcal{C}$-quasisupermodular function must have convex upper contour sets (in other words, that it is quasiconcave in the standard sense). Note that this is borne out by the example $f(x_1, x_2) = x_1x_2$.

PROPOSITION B4: *Let $X \subset R^l$ be a convex sublattice and let $f: X \to R$ be a continuous, increasing, and $\mathcal{C}_i$-quasisupermodular function. Then $\Psi = \{x \in X : f(x) \geq M\}$ is a convex set.*

PROOF: By adapting the proof of Proposition B2, we can easily establish that f is *quasiconcave in direction v for all v with $v_i > 0$ for some i and $v_j < 0$ for some j. Suppose that $f(x) - f(x + v) > 0$. By the $\mathcal{C}_i$-quasisupermodularity

[27]There is an easy way to see that a quasiconcave function (in the standard sense) can be only countably discontinuous. First, note that there must be some t^* in the domain of the function such that the function is increasing for $t < t^*$ and is decreasing for $t > t^*$. Second, increasing and decreasing functions can have only countably many discontinuities.

of f, for any positive scalar s we have $f(x+sv_+) - f(x+v+sv_+) > 0$, and by $\mathcal{C}_j$-quasisupermodularity, we have

$$\begin{aligned} &f(x+sv_+ + sv_-) - f(x+v+sv_+ + sv_-) \\ &\quad = f(x+sv) - f(x+v+sv) > 0, \end{aligned}$$

which shows that f is *quasiconcave in direction v. Now consider two distinct points x' and x'' in Ψ. If $x'' > x'$ or $x' > x''$, it is clear that $f(\alpha x' + (1-\alpha)x'') \geq M$ for α in $[0,1]$ because f is increasing. So we assume that x' and x'' are not ordered, in which case we know that f is *quasiconcave in the direction $\bar{v} = x'' - x'$. Because we assume that f is continuous, it is also quasiconcave in the direction $\bar{v}$, which means that $f(\alpha x' + (1-\alpha)x'') \geq M$ for all α in $[0,1]$. *Q.E.D.*

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